Semester 3 MJC-04 Diff. eq.

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TYPE 3 Equations solvable for x.

Form: $x = f(y, p)$

Diff w.r. to y

$$\frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y}$$

i.e. $\frac{1}{p} = \frac{\partial f}{\partial y} \phi(y, p, \frac{dp}{dy})$.

To solve by using previous rule.

Ex (1) solve: $x = y - p^2$

Soln:- Given: $x = y - p^2$ — (1)

Diff (1) w.r. to y

$$\frac{dx}{dy} = \frac{dy}{dy} - \frac{dp^2}{dp} \times \frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = 1 - 2p \cdot \frac{dp}{dy}$$

$$\Rightarrow 2p\frac{dp}{dy} = 1 - \frac{1}{p}$$

$$\Rightarrow \frac{dp}{dy} = \frac{1}{2p}\left(\frac{p-1}{p}\right)$$

$$\Rightarrow dy = \frac{2p^2}{(p-1)} dp$$

Taking Integration

$$\Rightarrow \int dy = 2\int \frac{(p^2-1)+1}{p-1} dp$$

$$\Rightarrow y = 2\int \left(p+1 + \frac{1}{p-1}\right) dp$$

$$\Rightarrow y = 2\left[\frac{p^2}{2} + p + \log|p-1|\right] + c$$

$$\Rightarrow y = p^2 + 2p + 2\log(p-1) + c \quad \text{— (2)}$$

Hence, from (1),

$$\Rightarrow x = y - p^2$$

$$\Rightarrow x = \cancel{p^2} + 2p + 2\log(p-1) + c - \cancel{p^2}$$

$$\Rightarrow x = 2p + 2\log(p-1) + c \quad \text{— (3)}$$

The p-elimination of (2) and (3) gives the general solution.

Ex (2) Solve: $y = 2px + y^2p^3$

Soln:- Given $y = 2px + y^2p^3$ — (1)

$$\Rightarrow 2px = y - y^2p^3$$

$$\Rightarrow 2x = \frac{y}{p} - y^2p^2 \quad \text{— (2)}$$

Diff (1) w.r. to y

$$2\frac{dx}{dy} = \frac{1}{p}\frac{dy}{dy} + y\frac{d\frac{1}{p}}{dy} - \left\{p^2 \cdot \frac{dy^2}{dy} + y^2 \cdot \frac{dp^2}{dy}\right\}$$

$$\Rightarrow \frac{2}{p} = \frac{1}{p} + y \cdot \left(-\frac{1}{p^2}\right)\frac{dp}{dy} - p^2 \cdot 2y - y^2 \cdot 2p \cdot \frac{dp}{dy}$$

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$$\Rightarrow \frac{2}{p}-\frac{1}{p}=\frac{-y}{p^2}\frac{dp}{dy}-2p^2y-2py^2\cdot\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p}+2p^2y+\frac{dp}{dy}\left(\frac{y}{p^2}+2py^2\right)=0$$

$$\Rightarrow \left(\frac{1+2p^3y}{p}\right)+\left(\frac{y+2p^3y^2}{p^2}\right)\frac{dp}{dy}=0$$

$$\Rightarrow (1+2p^3y)\cdot\left\{\frac{1}{p}+\frac{y}{p^2}\frac{dp}{dy}\right\}=0$$

$$\therefore \frac{1}{p}+\frac{y}{p^2}\cdot\frac{dp}{dy}=0$$

$$\Rightarrow \frac{y}{p^{\cancel{2}}}\frac{dp}{dy}=\frac{-1}{\cancel{p}}$$

$$\Rightarrow -\frac{dp}{p}=\frac{dy}{y}$$

Taking Integration.

$$\Rightarrow \int\frac{dy}{y}+\int\frac{dp}{p}=0$$

$$\Rightarrow \log y+\log p=\log c$$

$$\Rightarrow \log(py)=\log c$$

$$\Rightarrow py=c \Rightarrow p=\frac{c}{y}$$

Putting the value of p in (1)

$$y=2px+p^3y^2$$

$$\Rightarrow y=2x\cdot\frac{c}{y}+y^2\times\frac{c^2}{y^3}$$

$$\Rightarrow \boxed{y^2=2cx+c^3}$$ This is the general soln.

EX (3) Solve: $y^2\log y=xyp+p^2$

Soln: Given: $y^2\log y=xyp+p^2$

$$\Rightarrow pxy=y^2\log y-p^2$$

$$\Rightarrow x=\frac{y^2\log y}{py}-\frac{p^2}{py}$$

$$\Rightarrow x=\frac{y\log y}{p}-\frac{p}{y}$$

Ans ∴ $\log y=cx+c^2$

D) [Do Yourself]

EX (4)

Type 4 Equations in which either x or y is absent

~~W.R.:- If in a differential equation, x is absent~~

~~then it can be expressed in the form $y = f(p)$.~~

~~It is solved by using rule of Type 2 and vice versa~~

3.2 "Clairaut's Equation"

The equation of the form

$y = px + f(p)$ —— (I), where $p = \frac{dy}{dx}$ and $f(p)$ is any function of p is called as clairauts equation.

In order to solve (I), we diff. (I) w.r.to x.

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\cdot\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = \cancel{p} + \frac{dp}{dx}(x + f'(p))$$

Either, $\frac{dp}{dx} = 0$ or $x + f'(p) = 0$ —— (III)

$\Rightarrow dp = 0$

$\Rightarrow p = c$ —— (IV)

Putting $p = c$ in (I)

$\therefore y = cx + f(c)$ —— (V)

which is called ~~general or complete~~ Singular solution of given equation.

Remark (I) If we eliminate p between (1) and (2), we shall get a relation between x and y

(II) We see that the equations of clairauts form admit of two kind of solutions

(a) The complete solution containing one arbitrary constant

(b) The singular solution containing no arbitrary constant

(c) It has to be noted that the act of eliminating p between (I) and (III) as $y = px + f(p)$ and $x + f'(p) = 0$ is same as that of eliminating c between $y = cx + f(c)$ and $x + f'(c) = 0$.

Ex ① solve: $y = px + \frac{a}{p}$ and obtain the singular solution

Soln:- The given eqn

$$y = px + \frac{a}{p} \quad \text{— (i)}$$

It is in Clairaut's form

Hence, Its general solution is

$$y = cx + \frac{a}{c} \quad \text{— (ii)}$$

In order to find out the singular solution,

from (ii) $\Rightarrow cy = c^2x + a$

$\Rightarrow c^2x - cy + a = 0$, which is quadratic in c

and c has equal roots.

$\therefore B^2 - 4AC = 0$

$\Rightarrow (-y)^2 - 4 \times x \times a = 0 \Rightarrow y^2 - 4ax = 0$

$\therefore$ The singular solution is $y^2 - 4ax = 0$ Ans

Ex ② Solve: $(y - px)(p - 1) = p$

Soln: Given, $(y - px)(p - 1) = p$

$\Rightarrow y - px = \frac{p}{p-1}$

$\Rightarrow y = px + \frac{p}{p-1}$ — (i)

This is of the Clairauts form

Hence, its general solution is

$$y = cx + \frac{c}{c-1} \quad \text{— (ii)}$$

$\Rightarrow y = \frac{c^2x - cx + c}{c-1}$

$\Rightarrow c^2x - cx + c = cy - y$

$\Rightarrow c^2x - cx - cy + c + y = 0$

$\Rightarrow c^2x - c(x + y - 1) + y = 0$

Since, c has equal roots

$\therefore B^2 - 4AC = 0$

$\Rightarrow \{-(x+y-1)\}^2 - 4x \times y = 0$

$\Rightarrow (x+y-1)^2 - 4xy = 0$

$\therefore$ This is the singular solution.

Ex ③ solve: $y = px + \sqrt{a^2p^2 + b^2}$

Soln:- Do yourself $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ex ④ solve: $y = p(x - b) + \frac{a}{p}$

Soln:- Do yourself.

~~$y = cx - ac(1+c)$~~ $y = cx - bc + \frac{a}{c}$

~~Gen~~ Singular soln. $(x-a)^2 + 4ay = 0$

$y^2 = 4a(x - b)$